Rational Dimensia

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Rational Dimensia is a comprehensive examination of three questions: What is a dimension? Does direction necessarily imply dimension? Can all physical law be derived from geometric principles? A dimension is any physical quantity that can be measured. If space is considered as a single dimension with a plurality of directions, then the spatial dimension forms the second of three axes in a Dimensional Coordinate System (DCS); geometric units normalize unit vectors of time, space, and mass. In the DCS all orders n of any type of motion are subject to geometric analysis in the space-time plane. An nth-order distance formula can be derived from geometric and physical principles using only algebra, geometry, and trigonometry; the concept of the derivative is invoked but not relied upon. Special relativity, general relativity, and perihelion precession are shown to be geometric functions of velocity, acceleration, and jerk (first-, second-, and third-order Lorentz transformations) with v^2 coefficients of $n! = 1$, 2, and 6, respectively. An nth-order Lorentz transformation is extrapolated. An exponential scaling of all DCS coordinate axes results in an ordered Periodic Table of Dimensions, a periodic table of elements for physics. Over 1600 measurement units are fitted to 72 elements; a complete catalog is available online at EPAPS. All physical law can be derived from geometric principles considering only a single direction in space, and direction is unique to the physical quantity of space, so direction does not imply dimension.

I. INTRODUCTION

What is a dimension? Does, as Edwin Abbott puts forth in his 1884 novella *Flatland*, "direction imply dimension"?[1] Can all physical law be derived from geometric principles as Minkowski predicts?[2] To answer these questions one must first consider the myriad of myriad of different measurements mankind has used throughout history. Which are unique dimensions and which are redundant? Not only are there scores of dimensia (*n*, *pl* 1: the plural of sets of dimensions, i.e., money, monies): SI, mks, cgs, etc., but within each system there are numerous quantities that express the same dimensionality. For length one could measure a wavelength, or a path length, or a radius, or a diameter—and in the SI system one could even measure an energy and choose to call it a temperature. It is this dimensia (*n*, *pl* 2: the minutiae of dimensions) that frustrates the researcher of geometric relations between the dimensions. It is the thesis of this article that these dimensia can be rationalized, and with that rationalization, the underlying geometric relations are revealed.

By characterizing space as a single dimension with a plurality of directions and building on its one-to-one equivalence with time in the space-time plane, a Dimensional Coordinate System (DCS) composed of the three physical dimensions mass m, space d, and time t can be constructed. Within that construction, kinematic and relativistic effects are examined and expanded beyond current knowledge, based purely on geometric principles and the constancy of the speed of light. Validation of m as a third axis is provided in the form of an exponential scaling of the coordinate axes, resulting in an ordered Periodic Table of Dimensions. Within this table over 1600 units of measurement have been fit to 72 elements, and both arithmetic and geometric methods of dimensional analysis can be applied to all. The power and efficiency of dimensional analysis make it an extremely valuable tool for the sciences.[3–5] It is the proposition of this author that all physical measurements can be expressed and analyzed in the terms of the Periodic Table of Dimensions.

II. CONTRACTION OF THE SPACE METRIC

> *Pluralitas non est ponenda sine necessitate: Plurality should not be posited without necessity.*
>
> -Occam's Razor

A. Physical metrics

Consider the Pythagorean distance formula

$$x^2 + y^2 = d^2, \tag{1}$$

illustrated in Fig. 1(a), which gives the magnitude of the space metric d in two directions. Given three spatial axes, this Euclidean metric is expanded to express the magnitude

$$x^2 + y^2 + z^2 = d^2 \tag{2}$$

of a spatial three-vector. The spatial three-vector is shown in comparison to the spatial two-vector in Fig. 1(b). The scalar magnitude d^2 is the starting point of most commonly cited discussions of relativity.[2,6–11] Theoretically, the space metric can be expanded to include any number of components, but practical geometric representations of four or more mutually orthogonal components seem to be nonexistent.

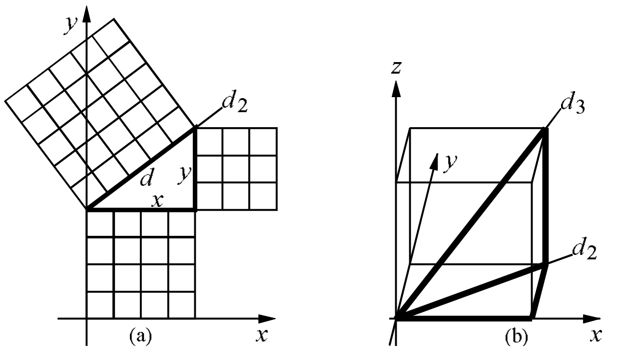

FIG. 1: The space metric d in (a) two and (b) three directions.

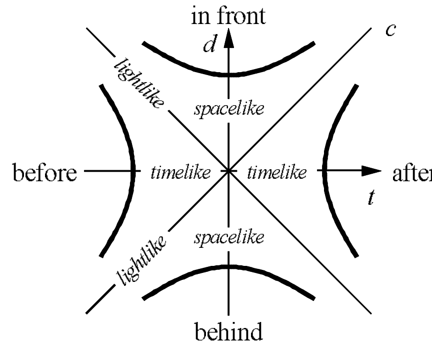

FIG. 2: The timelike and spacelike hyperbolae of space-time separated by lightlike asymptotes.

In consideration of physical phenomena, the distance given by the space metric can be set equal to the distance light travels in one second, i.e., 3×10^8 meters. Algebraic manipulation of the dimensional relation for light speed $c = d/t$ gives

$$d^2 = c^2 t^2 \tag{3}$$

so that

$$x^2 + y^2 + z^2 = c^2 t^2, \tag{4}$$

as presented by Einstein and others.[8,9,12,13] Note that in Eq. (3), t^2 could just as easily be the isolated variable, leaving d^2/c^2 expressed in square seconds.[14,15]

Collecting both the three-component term for space and the term for time from Eq. (4) on either side of the equal sign gives two equally valid expressions:

$$x^2 + y^2 + z^2 - c^2 t^2 = 0 \tag{5a}$$

and

$$c^2 t^2 - x^2 - y^2 - z^2 = 0. \tag{5b}$$

These two forms impart antisymmetric metric signatures of $(+,+,+,-)$ and $(+,-,-,-)$, respectively, to the components.[2,6,9,10,12,13,16] For the purposes of this discussion, the $(+,-,-,-)$ metric is employed unless otherwise noted; both are prevalent throughout the literature.[2,6,7,9–24]

When these expressions are equal to zero, as in Eqs. (5a) and (5b) and as shown by the line labeled c in Fig. 2, the interval is called lightlike.[6,12,25,26] Only phenomena propagating at the speed of light can take place over lightlike intervals. Intervals other than those accessible only at light speed can be considered by letting the result of Eqs. (5a) or (5b) vary. Designating the interval given by these expressions as s, and limiting the discussion to infinitesimal local frames of reference, gives the scalar length of a worldline in the familiar form of the space-time metric:

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = ds^2. \tag{6}$$

Whether infinitesimal or not, when s^2 is greater than zero (positive), the interval is called timelike and can be anywhere in the areas labeled timelike in Fig. 2; when s^2 is less than zero (negative), the interval is called spacelike and can be anywhere in the areas labeled spacelike in Fig. 2.[2,6,12,18,25–28] Physical interactions can occur in timelike intervals; only physical separation can occur over spacelike intervals.

Consideration of a spacelike interval $c^2t^2 - x^2 - y^2 - z^2 = -s^2$ implies imaginary coordinates. Some have suggested[2,9,10,22] that this imaginary coordinate originates as a coefficient of either the term for space, i.e., $(id)^2 = i^2(x^2 + y^2 + z^2)$, or the term for time, i.e., $(ict)^2$. Others, such as Misner, Thorne, and Wheeler point out, "This imaginary coordinate was invented to make the geometry of space-time look formally as little different as possible from the geometry of Euclidean space."[6,25] This is demonstrated by the similarity of the two general forms for a spatial two-vector in imaginary space, $(ct)^2 + (id)^2 = s^2$ and $(ict)^2 + (d)^2 = s^2$, to the Pythagorean distance formula in Eq. (1).

But space-time is not imaginary; space-time is real and hyperbolic.[2,10,13,16,19,26–30] Intervals are not given by the sum of squares, but by the difference of squares—the squares of real numbers. The only imaginary term occurs in s, which is imaginary only when the difference of real coordinates is negative, i.e., spacelike, resulting in a 90-degree rotation of the hyperbolic axes. Figure 2 illustrates the hyperbolic asymptotes of time and space, and their relation to the asymptotic light speed axis. When the difference of the squares is negative, the result is $-(s)^2$, the root of which is is. This does not result from either the term for space or the term for time being less than zero, but from the ratio of d/t referred to in Eq. (3) exceeding the value of light speed c.

Equation (6) is the rectangular form of the space-time metric, which is equivalent to Eq. (5a) when the $(+,+,+,-)$ metric is employed.[2,6,7,9–14,16,19,20,22] Other forms of the space-time metric frequently found in the literature include the metric tensor $g_{\mu\nu}$ introduced by Einstein in the general theory. This is commonly presented in matrix form,

$$ds^2 = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (7)$$

or with the shorthand notation diag$(+1, -1, -1, -1)$, with all signs reversed in the case of the $(-, +, +, +)$ metric.[6,10,12,17,18,21–23] (By convention time is assigned to the zeroth of a 0-3 index here, rather than the fourth of 1-4.) In tensor notation the interval of the space-time metric is given by the relation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \qquad (8)$$

where the superscripts μ and ν are indices of 0, 1, 2, 3 rather than exponents, and Einstein's summation convention is applied.[6,7,9–12,20,21,31,32]

Two significant polar forms of the space-time metric also warrant mention. The first is the Schwarzschild metric equation

$$\begin{aligned} ds^2 = c^2 dt^2 (1 - 2m/r) - \frac{dr^2}{(1 - 2m/r)} \\ -r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \end{aligned} \qquad (9)$$

where r, θ, and ϕ are spherical coordinates, and $m = GM/c^2$ where G is the gravitational constant and M is the total mass of the system; the condition where $r = 2m$, so that the term $1 - 2m/r$ goes to zero, is the Schwarzschild radius; an object of mass m smaller than its Schwarzschild radius is a black hole.[7,33–35] The second polar form that should be mentioned is

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \qquad (10)$$

where r, θ, and z are cylindrical coordinates of flat Minkowski space-time in string theory.[33]

No matter the form, the scalar interval given by s^2 is the space-time metric. In a grammatical sense all these forms are synonymous; they are many ways of saying the same thing. The s^2 resulting from any one form of the space-time metric is the same s^2 given by any other form for the same interval, and when any transformation s' that conforms to the postulates of relativity is applied, $s^2 = (s')^2$. The space-time metric expresses space-time increments equivalently with respect to any inertial reference frame, a property called invariance. While direction can be determined from initial values, the space-time metric gives the scalar length of a worldline without regard to direction, while the linear velocity vector, with tangent d/t, always gives the direction.

B. Space-time equivalence

In the special theory of relativity, Einstein sets space and time equal, $t = x = y = z = 0$, treating t, x, y, and z as equivalents, to the point of melding them into a single space-time continuum.[8] Both Minkowski and Weyl also contend that time is on equal footing with the three directions of space,[2,11] which permits identical treatment of the four coordinates x, y, z, and t. And in Edwin Abbott's *Flatland*, A. Square, the two-dimensional narrator of the story, remarks to Lord Sphere of three-space that "dimension implies direction and measurement."[1] Considering the context, this last statement begs the question, exactly how many directions does dimension imply? Two, as in Flatland? Three, as in Lord Sphere's world?

Typically, relativistic systems[6–9,12,13,18,25,33,36] consist of the three directions, x, y, and z. When three directions are given, almost without exception, y and z are simply set to zero and all events are considered to occur along the x-axis. As can be seen from the Lorentz transformations in three directions,

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (11)$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (12a)$$

$$y' = y \qquad (12b)$$

$$z' = z, \qquad (12c)$$

considering y and z has absolutely no effect on the physics whatsoever.[8–10,12,37] The limitation lies in the difficulty of geometrically representing four directions, all at right angles to each other. Ohanian[6] addresses this limitation by simply omitting the z-coordinate in constructing the light cone $t^2 - x^2 - y^2 = 0$, while Muller[38] finds it revealing to assign y to the spin axis and proceed in t, x, and z.

For the system of one spatial direction the space axis is simply d, the composition of the spatial axes of a multidirectional system as defined in Eqs. (1) and (2) and illustrated in Fig. 1. Cook says it most succinctly when he notes that it is the proportion of $d\ell$, a composition of the x, y, and z directions, to time that places the speed of light at the invariant value c in all directions, and gives the space-time metric in one spatial direction as $ds^2 = -c^2 dt^2 + d\ell^2$, albeit with the $(-, +, +, +)$ metric.[7] Some, including Minkowski, disregard y and z for the sake of simplicity.[2,12,24,26,27] Others, including Pierseaux, who quotes Poincaré extensively on this subject, point to the arbitrariness of the x-axis and use the convention of assigning to space a single direction defined by the linear velocity vector.[16,36] In the end it all boils down to one direction in space at any one instant in time.

The addition of directions to the dimension of space conserves the dimensionality of each direction in the nonlinear proportion of the sum of squares. Since all forms of the space-time metric are equivalent, consideration of the rectangular form in Eq. (6) is equivalent to consideration of any of the other forms, including those presented in Eqs. (7) through (10). Extrapolating the pattern given by the rectangular forms for spatial dimensions of one, two, and three directions gives $d^2 = x^2, = x^2 + y^2, = x^2 + y^2 + z^2, = x^2 + y^2 + z^2 + \cdots$. The relationship of the terms for space and time are such that the sum of the squares of the spatial directions, i.e., Eq. (2), is proportionate to the square of the single term for time c^2t^2. If x, y, and z were each individually of the same physical nature as time, then the proper proportions would be $ct : x$, $ct : y$, and $ct : z$ rather than $ct : d$. Geometrically, the composition of directions is Euclidean, but the composition of space-time is hyperbolic. Numerically, this inequivalence of direction to time varies with the number of directions considered and reaches the following maxima when distances in each of the directions considered are set equal to each other:

$$ct : d(x) = 1 : 1 \tag{13a}$$

$$ct : d(x, y) = 1 : \frac{1}{\sqrt{2}} = 1 : 0.707 \tag{13b}$$

$$ct : d(x, y, z) = 1 : \frac{1}{\sqrt{3}} = 1 : 0.577 \tag{13c}$$

$$ct : d(w, x, y, z) = 1 : \frac{1}{\sqrt{4}} = 1 : 0.5 \ldots \tag{13d}$$

It is a fundamental proposition of this paper that space is a single dimension, equivalent to time, with three (more or less) directions. Corollary to that, direction is not equivalent to dimension, as demonstrated in the Lorentz transformations, Eqs. (11), and (12a) through (12c). Geometrically, the space-time metric represents the length of a worldline in a two-axis coordinate system composed of a single time axis and a single space axis. The space axis is a Pythagorean composition of the three (more or less) spatial directions x, y, and z, as in Eq. (2), aligned along the axis of linear motion. Only the axis parallel to linear translation is affected in the same manner as time. If space is considered as a single dimension of equal standing to time, with three (more or less) directions, the hyperbolic identity of distance $x^2 - y^2 = s^2$ holds true for a worldline in two-dimensional space. Brill and Jacobson[28] provide a very enlightening discussion of this $x^2 - y^2 = s^2$ geometry that also includes a difference of squares geometric proof in the manner of the Pythagorean prooof illustrated as $x^2 + y^2 = d^2$ in Fig. 1. This is the geometry of relativity.

The composition of spatial directions is independent of any absolute direction in an inertial reference frame considered to be at rest. As Einstein notes in one popular exposition on relativity, “the most careful observations have never revealed such anisotropic properties in terrestrial physical space, i.e., a physical nonequivalence of different directions.”[9,39] In other words, no experiment has been done that can identify an x direction independently of the y or z directions. However, when involving kinematics, special relativity shows that there exists a geometrically preferred direction along the axis of uniform linear translation, also known as the velocity vector. Convention assigns this to the x-axis in the Lorentz transformations for space, Eqs. (12a) through (12c), but by Lorentzian definition the dynamics of space and time are not affected along the y and z axes only because motion does not occur along the y and z axes, i.e., $y = 0, z = 0$. But if no particular direction can be identified as the x direction, then applying the velocity vector specifically to the x-axis is completely arbitrary.[16,36]

Relativistic contraction in space occurs only in the direction of linear motion. Neither x, nor y, nor z matters—only the direction of velocity v contracts. Then any space-time metric taken in the direction d, defined by the velocity vector, can be said to be in contracted form. The contraction is also grammatical in nature in that the term for space is “shortened by omission” (Webster’s: contraction) of extraneous directions. The space-time metric then becomes a simple two-vector in a space-time plane that conforms to hyperbolic plane geometry as the magnitude

$$s^2 = c^2t^2 - d^2 \tag{14}$$

with a slope given by the velocity

$$v = \frac{d}{t}. \tag{15}$$

Pythagorean contraction of the space metric should not be confused with a tensor contraction of the space-time metric as discussed by Jackson and Ohanian, in that the rank of the remaining space-time matrix is only reduced by one.[6,12] This Pythagorean contraction derives its name purely from the physical phenomenon associated with it. The space metric d, on its own, is reduced in rank from a 3^2 matrix to a 3^0 matrix, and accordingly some directional information, i.e., x, y, z, but not v, is lost. However, no dimensional information is lost. The full tensor contraction of the space-time metric from a 4^2 matrix to a 4^0 matrix results in the scalar s^2; d and t have direction, s does not.

C. Orientation

When using the contracted space metric, i.e., Eq. (2), to express the velocity vector in Eq. (15), some rational tenets of geometric analysis require revisiting. First and foremost, the slope of a line is its rise over run. If v is to be the slope of a line as in Eq. (15), then the rise is d, and the run is t. The general geometric statement, with the slope given as m in the xy-plane, is

$$m = \frac{y}{x}, \tag{16}$$

or as expressed in slope-intercept form, where b is the y-intercept:

$$y = mx + b. \tag{17}$$

Substituting t for x, d for y, and v for m in Eq. (16) gives Eq. (15), also equivalent to Eq. (17) when b is set to zero. Another frequently invoked property of the velocity vector is that its angle θ from the t-axis is given by the inverse tangent of the velocity, or conversely[25,26,30,40]

$$tan\,\theta = \frac{d}{t}, \tag{18}$$

and

$$tan\,\theta = \frac{y}{x}. \tag{19}$$

Given Eqs. (16), (18), and (19), it seems more rational to propose that t goes to the x-axis and d goes to the y-axis than the contrary.

Traditionally, for reasons unspecified, time has been assigned to the y-axis.[2,6,12,15–17,19,25–30,41,42] Some exceptions have been noted, but no rationale seems to be given for these exceptions.[42–44] However, in a chapter on kinematics in one dimension, Ohanian conducts an analysis of worldlines that includes approximations of a curved worldline as the derivative $v = dx/dt$, which is a tangent line with slope $(x_2 - x_1)/(t_2 - t_1) = \Delta x/\Delta t$ taken between any two infinitesimally separated points on the worldline.[45] In order to show the algebraic relation of the slope, the tangent, and the derivative to the space-time plane, a.k.a. the xy-plane, the t-axis is made the horizontal axis. This is the standard that should be practiced with the same implications as maintaining the orientation of the y-axis to the x-axis in any Cartesian system.

Any system can consider itself to be at rest. Any stationary system, i.e., $v = 0$, will have a velocity vector parallel to the resting system's t-axis. The direction of the t-axis defines the state of a system at rest. Since any system may be considered at rest, every system has a defined t-axis. When $v = 0$ the space-time metric in Eq. (14) and the t-axis are parallel because only the t term contributes to the s term; every measurement takes some finite amount of time. Any system in uniform motion, i.e., $v > 0$, will have a velocity vector at an angle to the t-axis of the resting system. The velocity vector in the resting system is parallel to the t-axis of the system with velocity v; any system can consider itself at rest.

The space metric of the system with velocity v, in its proper orientation, is always taken to be in the direction of motion; the t-axis of the (resting) system measuring v is by definition at an angle of $tan^{-1}v$ clockwise from the direction of motion. The d-axis is always perpendicular to the t-axis.[18,21] Making d the functional axis, secondary to time, in a right-handed system, facilitates the treatment of space-time in accordance with all the accepted rules and methods of geometric analysis in any xy-plane.

D. The mass axis

Contraction of the space metric reduces a four dimensional problem to one of two dimensions. As discussed in Sec. II A, practical geometric representations of four or more components orthogonal to each other seem to be nonexistent. However, the representation of three orthogonal components as x, y, and z is very familiar. Contracting the space metric not only simplifies the space-time metric, but also results in a free axis within the confines of familiar three-vector mathematics. It would be nice if there were a third vector quantity to be represented by that free z-axis. Preferably this quantity would be dimensional so as to obey certain, yet to be determined, fundamental physical laws of dimensions also applicable to space and time. It should also be related to space-time in a right-hand manner just as space is related to time. Also, there must be some transformation, such as Eq. (3), that relates units on this new axis to the geometric units on the t- and d-axes.

The Planck units provide the transformation necessary to express mass as a geometric quantity on the third axis. The Planck units for time t_{P}, space d_{P}, and mass m_{P} are

$$t_{\mathrm{P}}^2 = \frac{\hbar G}{c^5} \tag{20a}$$

$$d_{\mathrm{P}}^2 = \frac{\hbar G}{c^3} \tag{20b}$$

$$m_{\mathrm{P}}^2 = \frac{\hbar c}{G}, \tag{20c}$$

where $\hbar$ is Planck's constant h divided by 2π, and G is the gravitational constant. The relations between these are

$$ct_{\mathrm{P}} = d_{\mathrm{P}} = \frac{m_{\mathrm{P}} G}{c^2}, \tag{21}$$

so that $d_{\mathrm{P}}^2/t_{\mathrm{P}}^2 = c^2$ as given in Eq. (3), and the unit vectors relating a three-axis system of physical dimensions on the Planck scale are[17]

$$\mathbf{t}_{\mathrm{P}} = ct_{\mathrm{P}} \tag{22a}$$

$$\mathbf{d}_{\mathrm{P}} = d_{\mathrm{P}} \tag{22b}$$

$$\mathbf{m}_{\mathrm{P}} = \frac{m_{\mathrm{P}} G}{c^2}, \tag{22c}$$

which go to the axes traditionally labeled as x, y, and z, respectively. The Pythagorean magnitude d of the space metric given by $d^2 = x^2 + y^2 + z^2$ is then analogous to the magnitude r of the Pythaogrean dimensional metric:

$$r^2 = (ct)^2 + (d_x^2 + d_y^2 + d_z^2) + \left(\frac{mG}{c^2}\right)^2; \tag{23}$$

whereas the Minkowskian dimensional metric is of the form

$$s^2 = t^2 - d^2 \pm m^2, \tag{24}$$

with the $\pm$ nature of m to be left to future explorations. All of the rules applying to three-directional geometric analysis, as well as the rules applying to dimensional analysis, are equally applicable in this three-dimensional system of coordinates.

III. THE DIMENSIONAL COORDINATE SYSTEM

The whole universe is seen to resolve itself into similar world-lines, and I would fain anticipate myself by saying that in my opinion physical laws might find their most perfect expression as reciprocal relations between these world-lines.

-H. Minkowski[2]

A. The prime dimensions

Physical units can be assigned to the coordinate axes by the following relations, when the measurements of t, d, m, and the physical constants c and G are in SI units:

$$\mathbf{t} = ct \text{ meters} \tag{25a}$$

$$\mathbf{d} = \sqrt{d_x^2 + d_y^2 + d_z^2} \text{ meters} \tag{25b}$$

$$\mathbf{m} = \frac{Gm}{c^2} \text{ meters.} \tag{25c}$$

These relations express the scale for the physical interpretation of geometric analyses in the Dimensional Coordinate System (DCS). The quantities $\mathbf{t}$, $\mathbf{d}$, and $\mathbf{m}$ are unit vectors analogous to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively, in a Cartesian coordinate system. Using each of the fundamental dimensions as the unit dimension, three different scale factors can be derived, as shown in Table I along with the Planck scale. Each of these physical scale factors is equally valid and will lead to the same conclusions, as will any proper interpolation between them. Therefore, all of the following sets of physical relations are referred to as geometric units.[17] If there is a need, the unit axis, i.e., $\mathbf{t} = 1$, $\mathbf{d} = 1$, or $\mathbf{m} = 1$, can be so designated.

There is a unique geometric description in the Dimensional Coordinate System for every measurable quantity. The graph of each unit quantity is a straight line, either coincident with or parallel to its coordinate axis, with each unit vector mutually perpendicular to the others as shown in Fig. 3. Designating t as the unit axis, i.e., 1

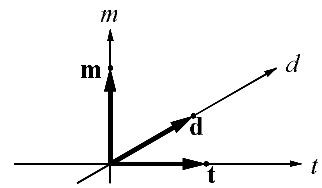

FIG. 3: The mutually perpendicular unit vectors of time $\mathbf{t}$, space $\mathbf{d}$, and mass $\mathbf{m}$ on their respective t-, d-, and m-axes.

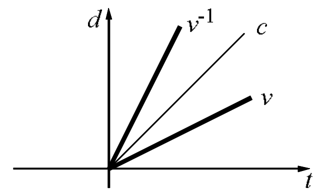

FIG. 4: Light speed c, and the velocity v and inverse velocity v^{-1} vectors in the space-time plane.

unit on the t-axis equals 1 second, a measurement of 1 second has a geometric description 1 unit long and parallel to the t-axis; a measurement of 3×10^8 meters has a geometric description 1 unit long and parallel to the d-axis, perpendicular to the t-axis; and a measurement of 4×10^{35} kilograms has a geometric description 1 unit long and parallel to the m-axis, perpendicular to both the t- and d-axes. Other magnitudes of mass, space, and time are described by scaling of the unit vectors.

B. First order motion

Consideration of the special case of uniform linear motion requires examination of the first-order rate of change of position, velocity v. The geometric description of velocity is a straight line of slope v with respect to the t-axis as shown in Fig. 4 and given by Eq. (15); $v = c = 1$ describes light speed. Qualitatively, with d as a function of v,

$$d(v) = vt. \tag{26}$$

Quantitatively, d as a function of constant velocity is also given by Eq. (26).

Both the space-time metric and the Lorentz transformations describe the length of the velocity vector with respect to the resting system. This is the basis of the invariance of s^2 with respect to the Lorentz transformations. From Eqs. (14) and (15),

$$c^2 - v^2 = \frac{s^2}{t^2}, \tag{27}$$

and therefore

$$1 - \frac{v^2}{c^2} = \left(\frac{s}{ct}\right)^2. \tag{28}$$

TABLE I: Geometric units of the Dimensional Coordinate System (DCS).

Units	time (t)	space (d)	mass (m)	Planck
seconds (s)	1	3.3×10^{-9}	2.5×10^{-36}	5.4×10^{-44}
meters (m)	3.0×10^{8}	1	7.4×10^{-28}	1.6×10^{-35}
kilograms (kg)	4.0×10^{35}	1.3×10^{27}	1	2.2×10^{-8}
Planck^{-1}	1.9×10^{43}	6.2×10^{34}	4.6×10^{7}	1

When compared to the γ-factor from the Lorentz transformations

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}, \tag{29}$$

it is seen that $\gamma = s/ct$, so that γ is dimensionless and normalized to the unit time. Remember that in Eq. (3) the choice to express t in light-seconds as a function of ct instead of d in seconds as a function of d/c was arbitrary. The velocity vector is a timelike interval; $v = c$ is lightlike. As discussed in Sec. II C, the velocity vector in the resting system is the t-axis of the system with velocity v.

The geometric description of inverse velocity v^{-1} is a straight line with slope v^{-1} with respect to the t-axis, as shown in Fig. 4 and given by the relation $v^{-1} = t/d$, the transpose of t and d in Eq. (15). Qualitatively, with d as a function of v^{-1},

$$d\left(v^{-1}\right) = \frac{t}{v^{-1}}. \tag{30}$$

Quantitatively, d as a function of constant inverse velocity is also given by Eq. (30).

The inverse velocity vector has the same length, given by the space-time metric or the Lorentz transformations, as the velocity vector. The inverse velocity vector of the resting system is a spacelike interval that forms the d-axis of the system with velocity v. Figure 5 shows the resting coordinate system of A with the velocity and inverse velocity vectors of a moving system B forming the t- and d-axes, respectively, from which the contracted coordinate grid of B, moving with velocity v_{B}, is constructed. This is the geometric description of the Lorentz transformations.[9,37,46]

To construct A's resting coordinate grid, lines parallel to the d-axis intersect the t-axis at unit intervals, and lines parallel to the t-axis intersect the d-axis at unit intervals. The resting system A will measure a contraction in the moving system B of magnitude γ from Eq. (29). For A to measure what B calls a unit interval, the moving unit must be γ^{-1} long in resting units to result in a measurement of exactly one unit. The moving unit in B is defined by A as longer than its own resting unit, so that after a contraction of magnitude γ is accounted for, they are equal. To construct B's coordinate grid, the axes for B's coordinate system are coincident to A's velocity and inverse velocity vectors for B's motion (v_{B} and v_{B}^{-1} are B's t- and d-axes, respectively). Parallel gridlines

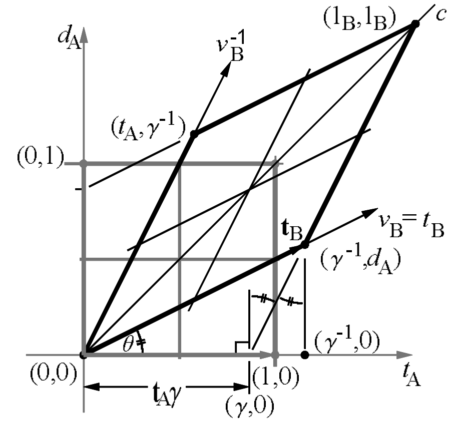

FIG. 5: The resting coordinate system of A, and the Lorentz transformed coordinate system of B with velocity v_{B}. The oblique triangle with vertices (0,0), (γ,0), and $(\gamma^{-1}, d_{\mathrm{A}})$ has one side formed by the B-unit interval $\mathbf{t}_{\mathrm{B}}$ and a second by B's contracted measurement of the A-unit interval $\mathbf{t}_{\mathrm{A}}\gamma$. The rhombi (0,0), (1,0), (1,1), (0,1) and (0,0), $(\gamma^{-1}, d_{\mathrm{A}})$, $(1_{\mathrm{B}}, 1_{\mathrm{B}})$, $(t_{\mathrm{A}}, \gamma^{-1})$ always enclose the same area.

intersecting B's t- and d-axes at intervals of γ^{-1} A-units define B's unit intervals and form its coordinate grid.

To verify this relation, the linear unit interval of $\mathbf{t}_{\mathrm{B}}$ from (0,0) to (γ^{-1},d_{A}) along B's t-axis is determined from the relationship

$$\mathbf{t}_{\mathrm{B}} = \frac{\mathbf{t}_{\mathrm{A}}}{\gamma \cos\,\theta}, \tag{31}$$

where $\mathbf{t}_{\mathrm{A}}$ is the resting unit interval from (0,0) to (0,1) and θ is the angle of the velocity vector from Eq. (18); $\mathbf{t}_{\mathrm{B}}$ is congruent to B's d-unit interval $\mathbf{d}_{\mathrm{B}}$. With B's unit intervals known, application of the law of sines to the oblique triangle (0,0), (γ,0), (γ^{-1},d_{A}) in Fig. 5 shows that

$$\mathbf{t}_{\mathrm{A}}\gamma = \mathbf{t}_{\mathrm{B}} \frac{\sin(\frac{\pi}{2} - 2\theta)}{\sin(\frac{\pi}{2} + \theta)}, \tag{32}$$

where $(\pi/2 - 2\theta)$ and $(\pi/2 + \theta)$ are the angles opposite the sides (0,0) (γ,0) and (0,0) (γ^{-1},d_{A}), i.e., $\mathbf{t}_{\mathrm{A}}\gamma$ and $\mathbf{t}_{\mathrm{B}}$, respectively; the triangles (0,0), (γ^{-1},0), (γ^{-1},d_{A}) and (γ,0), (γ^{-1},0) (γ^{-1},d_{A}) are similar triangles. In words, $\mathbf{t}_{\mathrm{A}}\gamma$, from B's resting perspective, is B's properly contracted unit interval of $(1 - v^2/c^2)^{1/2}$ moving A-units.

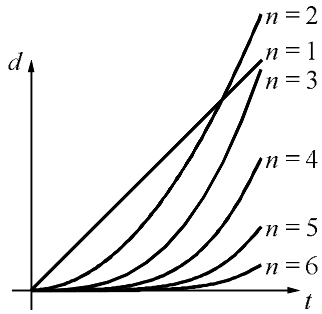

FIG. 6: Worldlines of first- through sixth-order curvature.

Since space and time have been defined as orthogonal quantities, it is worth noting that B's perpendicularity is not Euclidean from A's perspective. It is determined by the constancy of the speed of light; the angle $(\pi/4-\theta)$, between c and t, is always equal to the angle between c and d. This looks much the same way a circle viewed sideways turns into an ellipse - consider B's system to be at Euclidean right angles but viewed from an oblique angle. If the moving coordinate system is truly perpendicular and the unit distances are correct, then the area A_{B} of B's rhombus (0,0), $(\gamma^{-1},d_{\mathrm{A}})$, $(1_{\mathrm{B}},1_{\mathrm{B}})$, $(t_{\mathrm{A}},\gamma^{-1})$ in Fig. 5 and the unit area A_{A} of A's rhombus (0,0), (1,0), (1,1), (0,1) in the same figure should both be 1 square unit—A_{B} is invariant with respect to A_{A}. The length of each side of B's rhombus is equal to $\mathbf{t}_{\mathrm{B}}$ from Eq. (31). The acute angle between B's t- and d-axes is $2(\pi/4-\theta)=(\pi/2-2\theta)$. Then the area A_{B} of B's rhombus is $A_{\mathrm{B}}=\mathbf{t}_{\mathrm{B}}^2\sin(\pi/2-2\theta)=1$. The area A_{B} is Poincaré's *area invariant* as discussed by Pierseaux; Mermin reduces all of space-time to these rhombi of constant area.[16,27,47] Due to the relation of $\mathbf{t}_{\mathrm{B}}$ to γ as a function of v from Eq. (31) and thus to θ from Eq. (18), this is true for all values of θ derived from velocities less than c; the proof is left to the reader.

C. Greater orders of motion

Consideration of the general case of any motion requires the examination of second and greater orders of the rate of change of position. Change in velocity is acceleration a. The geometric description of acceleration is a line of second-order curvature with respect to the t-axis, as shown by $n=2$ in Fig. 6 and expressed by the relation[45]

$$\frac{d}{t^2}=\frac{v}{t}=a. \tag{33}$$

Qualitatively, d may be expressed as a function of a:

$$d(a)\propto at^2. \tag{34}$$

Quantitatively, from the origin with $v=0$, d as a function of constant acceleration is found by taking the antiderivative of both sides of Eq. (26) with respect to time and normalizing to the unit time vector. This results in the general form taught in first term physics,

$$d(a)=\frac{1}{2}at^2. \tag{35}$$

Change in acceleration is jerk j.[48–51] The geometric description of jerk is a line of third-order curvature with respect to the t-axis, as shown by $n=3$ in Fig. 6 and expressed by the relation

$$\frac{d}{t^3}=\frac{v}{t^2}=j. \tag{36}$$

Qualitatively, with d as a function of j,

$$d(j)\propto jt^3. \tag{37}$$

Quantitatively, d as a function of constant jerk, when starting from the origin with $v=0$, is found by taking the antiderivative of both sides of Eq. (35) and normalizing to the unit time vector, resulting in the general form[48]

$$d(j)=\frac{1}{6}jt^3. \tag{38}$$

Change in jerk is snap ψ, change in snap is crackle χ, and change in crackle is pop ϕ.[50] The geometric descriptions of snap, crackle, and pop are lines of fourth-, fifth-, and sixth-order curvature, respectively, as shown in Fig. 6 and expressed by the relations

$$\frac{d}{t^4}=\frac{v}{t^3}=\psi, \tag{39}$$

$$\frac{d}{t^5}=\frac{v}{t^4}=\chi, \tag{40}$$

$$\frac{d}{t^6}=\frac{v}{t^5}=\phi. \tag{41}$$

Qualitatively, with d as a function of snap, crackle, and pop,

$$d(\psi)\propto\psi t^4, \tag{42}$$

$$d(\chi)\propto\chi t^5, \tag{43}$$

$$d(\phi)\propto\phi t^6. \tag{44}$$

Quantitatively, in general form, d as a function of constant snap, crackle, or pop is given by the normalized antiderivatives, from the origin with $v=0$, of Eqs. (38), (45), and (46),

$$d(\psi)=\frac{1}{24}\psi t^4, \tag{45}$$

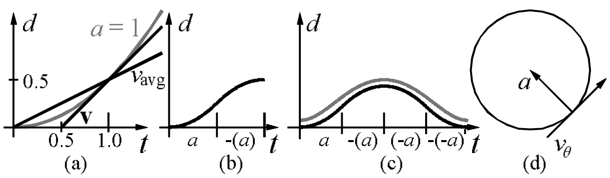

FIG. 7: (a) A worldline of acceleration with its tangent instantaneous velocity vector $\mathbf{v}$, and its secant average velocity vector v_{avg}. (b) A worldline with a period of acceleration a followed by an equal period of deceleration $-(a)$. (c) A worldline with the four permutations of linear acceleration a, $-(a)$, $(-a)$, and $-(-a)$, and a worldline of centripetal acceleration (a sine curve) offset slightly above it. (d) Circular motion with its v_θ and a vectors.

$$d(\chi) = \frac{1}{120}\chi t^5, \tag{46}$$

$$d(\phi) = \frac{1}{720}\phi t^6, \tag{47}$$

respectively. From these it is evident that the general, quantitative statement for d as a function of all orders n of any motion is given by the formula

$$d(n) = \frac{1}{n!}\frac{d}{\mathbf{t}^n}t^n. \tag{48}$$

Only for worldlines of zero and first orders, the geometric descriptions of distance and velocity respectively, are the average velocity v_{avg} and the instantaneous velocity $\mathbf{v}$ equal; $v_{\text{avg}} = \mathbf{v} = 0$ for the zeroth-order, and $v_{\text{avg}} = \mathbf{v}$ for the first.[45] For worldlines of second and greater orders v_{avg} and $\mathbf{v}$ no longer share the same geometric description. Figure 7(a) shows the relation of v_{avg} and $\mathbf{v}$ to the second-order curve for acceleration. The description of v_{avg} from $t = 0$ to any time t is given by

$$v_{\text{avg}} = \frac{d(n)}{t}, \tag{49}$$

while $\mathbf{v}$ is given by the slope of the tangent line to the curve of acceleration. Thus, the instantaneous velocity at any time t is given by the derivative of position as a function of acceleration. The instantaneous acceleration is given by the derivative of position as a function of jerk, and therefore the instantaneous velocity is given by the second derivative of position as a function of jerk. For orders up to and including the sixth, instantaneous jerk is given by the derivative of position as a function of snap, instantaneous snap is given by the derivative of position as a function of crackle, and instantaneous crackle is given by the derivative of position as a function of pop. Likewise, $\mathbf{v}$ is given by the third, fourth, and fifth derivatives of position as a function of snap, crackle, and pop respectively. Then $\mathbf{v}$ is given by the normalized derivative

$$\mathbf{v} = \mathbf{t}^{n-2}\frac{d^{n-1}}{dt^{n-1}}d(n). \tag{50}$$

The concept of average velocity has no physical meaning at any single instant in time, and the concept of position as a function of instantaneous velocity has no physical meaning for orders of n greater than one.

The significance of the average and instantaneous velocities of nth-order kinematic functions is in their proportion to each other. Equation (48) shows that a factor of $1/n!$ appears in the expression for $d(n)$, while differentiating Eq. (50) the prescribed number of times results in the cancellation of that factor, so that the proportion of v_{avg} to $\mathbf{v}$ is $1 : n!$; $\mathbf{v}$ is $n!$ times larger than v_{avg}. This is demonstrated graphically in Fig. 7(a) by the relative slopes of v_{avg} and $\mathbf{v}$. This proportion has a profound effect on the Lorentz transformations, where the method of measuring the velocity plays a crucial role for orders of n greater than one.

Decreasing velocity is deceleration $-(a)$. Deceleration is a permutation of acceleration during which the slope of the worldline is positive, but the change in slope is negative. The geometric description of deceleration is an inverted curve of second-order with respect to the t-axis, as shown in Fig. 7(b) and given by[45]

$$-\frac{d}{t^2} = -\frac{v}{t} = -(a). \tag{51}$$

Qualitatively, d as a function of deceleration is given by

$$d[-(a)] \propto -(a)t^2. \tag{52}$$

Quantitatively, d as a function of constant deceleration is given by

$$d[-(a)] = \frac{1}{2}[-(a)]t^2. \tag{53}$$

There are two descriptions for motion of this nature. A description of d in absolute terms is consistent with everything discussed to this point. With the addition of the operator $\pm$, relative motion in exact opposition to d can be uniquely described. Continuously repeating motions of these types, described either absolutely or relatively, encompass the class of physical phenomena known as harmonic motion. The absolute description is representative of phenomena such as wave propagation. The relative description is representative of phenomena such as springs, orbits, and, by extension, pendula.

Deceleration $-(a)$ is *not* negative acceleration $(-a)$; negative deceleration $-(-a)$ is not necessarily acceleration (a). In the strictest sense, deceleration is only that component of acceleration during which there is a reduction in velocity in the direction of d. During negative acceleration there is an increase in velocity, and during negative deceleration there is a decrease in velocity, but both are in the negative direction of d. During negative acceleration, both the slope and the change in slope are negative. During negative deceleration, the slope is negative while the change in slope is positive. In all cases, the term acceleration refers to a worldline of second-order with the slope and the change in slope of like sign. The term deceleration refers only to second-order worldlines with slope and change in slope of opposite sign. Upon substituting the term for negative acceleration into Eq. (53), one can see that deceleration $-(a)$ and negative acceleration $(-a)$ are described by the same curve. This is also true of acceleration (a) and negative deceleration $-(-a)$. By alternating Eqs. (35) and (53), a close approximation of the sine curve,

$$d = \sin\left(t - \frac{\pi}{2}\right), \tag{54}$$

with the appropriate scaling and offsets, is constructed.

The sine curve in Eq. (54) describes the worldline of an object in circular motion under centripetal acceleration a, from a plane-edge view, when d is parallel to the plane of the circular motion. The circular velocity remains constant, but the measured linear velocity along the plane of the motion varies as the direction of the velocity varies with respect to d. Circular motion on-edge describes a sine curve. Linear motion describes the second-order curves given in Eqs. (35) and (53). The two worldlines shown in Fig. 7(c) are not the same circle from two different perspectives. These are the worldlines of two different physical phenomena: linear acceleration and centripetal acceleration, each with $a = 1\ c/\mathrm{s}$.

Consideration of the case of circular motion requires the examination of angular velocity. If an event can be described as repeating in angular units θ, then it may be described as having an angular velocity ω. If an event can be described as repeating in temporal units, then it may be described as having a frequency f. The geometric description of continuous angular velocity is the sine curve shown parallel to the t-axis of the reference system in Fig. 7(c). The sine curve has amplitude $2r$ where r is the radius of curvature of the path of travel. Qualitatively, angular velocity and frequency are synonymous dimensions; both are given by

$$f \propto \omega = \frac{\theta}{\mathbf{t}}. \tag{55}$$

Quantitatively, d as a function of constant angular velocity is given by

$$d(\omega) = r - r\ \cos(\omega t), \tag{56}$$

and ω and f are related in the proportion $r : 2\pi r$; angular velocity is measured in radial units along the circumference per unit time, and frequency can be measured as circumferential units per unit time. Wavelength λ is a closely related measurement. A wavelength times a frequency gives a velocity, $\lambda f = v$. Wavelength is a spatial unit in $1:1$ proportion to the temporal unit f^{-1}, symmetric about the velocity c.

By taking the proportion of the circumferential path length ℓ, given by $\ell = 2\pi r$, to the period T, given by $T = 2\pi/\omega$, a constant velocity v_θ, where

$$v_\theta = \frac{\ell}{T} = \omega r \tag{57}$$

is obtained. Like all good constant velocities, this one too is subject to special relativistic effects.[52] Comparison of the two perspectives v_θ and $v = 0$, the resting system, reveals a difference in the measurements of ω. The system with v_θ measures ω_θ due to Lorentzian contraction in the proportion

$$\omega_\theta = \omega\sqrt{1 - \frac{v_\theta^2}{c^2}}, \tag{58}$$

while the system with $v = 0$ measures ω.

Circular motion requires a centripetal acceleration a with a linear velocity component v_θ as shown in Fig. 7(d). The geometric description of centripetal acceleration is the same line of second-order curvature with respect to the t-axis as shown in Fig. 7(a). A constant centripetal acceleration can be described by the sine curve Eq. (56), symmetric about a line parallel to the t-axis as shown in Fig. 7(c). Qualitatively, linear acceleration and centripetal acceleration are synonymous dimensions; both are given by Eq. (34). As described in Einstein's principle of equivalence, centripetal acceleration is indistinguishable from any other form of acceleration.[52,53] Quantitatively, the centripetal acceleration is related to the linear velocity component in square proportion:

$$\mathbf{v}_\theta^2 = ar. \tag{59}$$

Substituting Eq. (59) for v^2 in Eq. (29) gives

$$\gamma_\theta = \sqrt{1 - \frac{ar}{c^2}}. \tag{60}$$

The principle of equivalence is geometrically demonstrated by the correspondence of the curves for the different kinds of acceleration. Centripetal acceleration is a composition of linear accelerations from varying directions. Once corrected for, a worldline of the same acceleration from a single direction is geometrically indistinguishable. In practice this means that gravity is just as good a centripetal force as any other. Then the gravitational acceleration given by

$$a = \frac{GM}{r^2} \tag{61}$$

can be substituted into Eq. (60), to arrive at an expression

$$\gamma_g = \sqrt{1 - \frac{GM}{c^2 r}} \tag{62}$$

for a special relativistic effect in proportion to the gravitational field. However, as Einstein found, and others have discussed, this is only half of the gravitational effect measured; there is a missing factor of two.[10,20,34,35,53–57]

D. Factors of $n!$: a geometric description

For Einstein the development of the tensor field of general relativity revealed this factor of two, and the

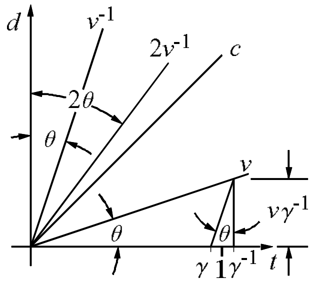

FIG. 8: The similar triangles generated by the Lorentz transformation, where v is the average velocity. The length of the base of the large triangle is γ^{-1}, and the side opposite the angle θ is $v\gamma^{-1}$. The sides opposite and adjacent the corresponding angle in the small triangle are $\gamma^{-1} - \gamma$ and $v\gamma^{-1}$, respectively. The hypotenuse of the small triangle is parallel to the inverse velocity vector, and $2v^{-1}$ is the inverse instantaneous velocity vector.

Schwarzschild metric depends on it. Schiff concludes that this factor of two arises from the sum of two separate terms, one for space and one for time. Strandberg derives it in a vector field by considering both tangential and radial vectors. However, in a dimensional field with only one direction, such extravagance is unnecessary. Instead, simple geometry is used to describe the same phenomenon revealed to Einstein in the tensor field. The physical reality of this factor of two has been quantitatively confirmed to a 99% precision by the redshift experiments of Pound and Rebka (1959) and Pound and Snider (1965) in a 22.6 meter tower. The body of physical evidence in support of the exact magnitude of these general relativistic effects is impressive and growing.[54,58]

The definition of velocity is the distance traveled per time traveled. In measuring the distance traveled over any period of acceleration, time must pass. The distance between two spatial points separated in time constitutes, by definition, the measurement of an average velocity v_{avg}; by definition the instantaneous velocity is instantaneous. However, as discussed in Sec. III C, the instantaneous velocity $\mathbf{v}$ is the velocity the relativist must contend with. Since $\mathbf{v}$ is the correct measure to use, but v_{avg} is the measure represented by the worldline, a modified Lorentz transformation must be derived to determine the contribution of the acceleration. Geometrically, this construction must result in perpendicular axes of $\mathbf{v}$ and $\mathbf{v}^{-1}$. If the slope of $\mathbf{v}$ is twice that of v_{avg} as discussed in Sec. III C, then the slope of its perpendicular must be half that of v_{avg}^{-1}.

The key is to express the Lorentz transformation in terms of $\mathbf{v}$. This is accomplished by taking advantage of the similar triangles discussed in Sec. III B and shown in Fig. 8. The length of the base of the large triangle is γ^{-1}. This is the side adjacent to the angle θ and coincident with the t-axis. The side opposite the same angle is given by $v_{\text{avg}}t$, where $t = \gamma^{-1}$, and also forms the base of the small triangle. The side opposite the angle θ of the smaller triangle is $\gamma^{-1} - \gamma$, so that v_{avg} of the smaller triangle is given by the ratio of the side opposite to the side adjacent:

$$v_{\text{avg}} = \frac{\gamma^{-1} - \gamma}{v\gamma^{-1}}. \tag{63}$$

This is the v_{avg} that, if doubled, will give the correct description for $\mathbf{v}$. For legibility $c^2 = 1$ is omitted in the following, and v^2 is considered dimensionless due to the implied division by c^2.

Substituting $2v = 2v_{\text{avg}} = \mathbf{v}$ for v_{avg}, the composition of $\mathbf{v}$ as a function of γ proceeds as follows:

$$\begin{aligned} 2v &= \frac{2v^2}{v} = \frac{1 - 1 + 2v^2}{v} \\ &= \frac{1 - (1 - 2v^2)}{v} = \frac{1}{v} - \frac{1 - 2v^2}{v} \\ &= \frac{\frac{1}{\sqrt{1-2v^2}}}{v\frac{1}{\sqrt{1-2v^2}}} - \frac{\sqrt{1 - 2v^2}}{v\frac{1}{\sqrt{1-2v^2}}} \\ &= \frac{\frac{1}{\sqrt{1-2v^2}} - \sqrt{1 - 2v^2}}{v\frac{1}{\sqrt{1-2v^2}}}; \end{aligned} \tag{64}$$

compare this to Eq. (63). From the final forms of Eqs. (62) and (64) it can be seen that γ as a function of a from Eq. (61), and properly corrected for $\mathbf{v} = 2v$ with c^2 reinserted, is actually given by

$$\gamma_a = \sqrt{1 - 2\frac{GM}{c^2 r}}. \tag{65}$$

The effects of Eq. (65) apply to space-time in any field of acceleration. An object static under a gravitational acceleration is affected equally to an object static with respect to a linear acceleration, i.e., in a rocket ship. When motion within the field of acceleration occurs, third-order motion called jerk occurs. Perihelion precession is a physical manifestation of jerk.

Einstein derived the magnitude of planetary perihelion precession $\Delta\phi$ as

$$\Delta\phi = \frac{24\pi^3 r_a^2}{T^2 c^2 (1 - e^2)} = 5.018 \times 10^{-7}\ \frac{\text{rad}}{\text{orbit}} \tag{66}$$

when values for Mercury are used.[10,52,55] This is often presented in terms of the product GM as

$$\Delta\phi = \frac{6\pi GM}{c^2 r_a (1 - e^2)} \tag{67}$$

by substituting Kepler's expression for the square of the period,

$$T^2 = \frac{4\pi^2 r_a^3}{GM}, \tag{68}$$

into Eq. (66). In his discussion on the perihelion precession, Strandberg gives

$$\Delta\phi = \frac{2\pi}{\sqrt{1-6\frac{GM}{c^2 r}}} - 2\pi \approx \frac{6\pi GM}{c^2 r} \tag{69}$$

as an approximation [although there does seem to have been a negative sign in the exponent omitted from Strandberg's Eq. (50)]. This is appropriate for a circular orbit of $e=0$, as the factor $(1-e^2)$ scales the result to the semi-latus rectum r_ℓ as a function of eccentricity.

To show the relationship of Einstein's approximation to the Lorentz contraction, beginning with Eq. (67) in radians per orbit, the number of radians given is the difference of the number of radians in a Euclidean orbit from the number of radians in a relativistic orbit. The number of radians in a Euclidean orbit is 2π. The number of radians in a relativistic orbit is greater than 2π. From Eq. (67), multiplying by $1/2\pi$ orbits per radian,

$$\frac{6\pi GM}{c^2 r_\ell}\,\frac{\text{rad}}{\text{orbit}} \times \frac{1}{2\pi}\,\frac{\text{orbits}}{\text{rad}} = 3\frac{GM}{c^2 r_\ell}\,\frac{\text{orbits}}{\text{orbit}}, \tag{70}$$

where r_ℓ is given by $r_\ell = r_a(1-e^2)$, leaves the orbital excess. This is the portion of the relativistic orbit in excess of the Euclidean orbit. The expression for the complete relativistic orbit is simply one Euclidean orbit plus the excess, or

$$1+\frac{\Delta\phi}{2\pi} = 1+3\frac{GM}{c^2 r_\ell}. \tag{71}$$

Keeping in mind that, prior to the advent of computers, approximation methods such as log tables, trig tables, and slide rules were common, Einstein's application of two popular approximations,

$$1+x \approx \frac{1}{1-x}, \tag{72}$$

and

$$1-\frac{1}{2}x \approx \sqrt{1-x}, \tag{73}$$

can be extracted.[10,20,52,57] The result is that

$$1+3\frac{GM}{c^2 r_\ell} \approx \frac{1}{\sqrt{1-6\frac{GM}{c^2 r_\ell}}}. \tag{74}$$

Two factors contributing to precession can be isolated in the discriminating term of the Lorentz factor on the right-hand side of Eq. (74),

$$6\frac{GM}{c^2 r_\ell} = 6\frac{GM}{c^2 r_a(1-e^2)}. \tag{75}$$

The first factor,

$$\frac{GM}{r_a} = v_{\text{orb}}^2, \tag{76}$$

contributes to precession due to constant velocity in a circular orbit. The second factor, $(1-e^2)$, contributes to precession due to the eccentricity of the orbit. These two motions produce jerk—jerk due to the change in direction of the acceleration as a circular orbit varies in position, and jerk due to a change in the magnitude of the acceleration as an elliptical orbit varies in its radius r. Schot gives the magnitude of the jerk generated by a central force proportional to distance as[48,49]

$$j = -\omega^2 v. \tag{77}$$

To establish the relationship of the jerk to the perihelion precession, the discriminating term, i.e., Eq. (75), in the central expression of Eq. (69) must be expressed in terms of Eq. (77) to arrive at the same precession predicted by the general theory.

Expressing ω in terms similar to those of the discriminant,

$$\omega^2 = \frac{GM}{r_a^3}, \tag{78}$$

and taking note of the fact that ω^2 must be equal to v_{orb} for the product in Eq. (77) to be equivalent to Eq. (76), it is found that Eq. (77) is in excess by a factor of v_{orb}/r_a^2. Furthermore, $v_{\text{orb}}/2\pi r_a = T$, so in terms of the first-order Lorentz transformation,

$$v^2 = \frac{j r_a T}{2\pi(1-e^2)} \tag{79}$$

for the third-order Lorentz transformation. Substituting this term into the center expression of Eq. (69) and using values for the planet Mercury gives

$$\frac{2\pi}{\sqrt{1-6\frac{j r_a T}{c^2\, 2\pi(1-e^2)}}} - 2\pi = 5.018\times10^{-7}\,\frac{\text{rad}}{\text{orbit}} \tag{80}$$

as the correct value for the perihelion precession from Eq. (66), in terms of the jerk. In the final form, the contributions of the circular component (2π) and the eccentric component $(1-e^2)$ are clearly visible in the determinant.

The geometric components (2π) and $(1-e^2)$ tie directly to the spatial component r_a, which in turn is the related measure of d in the Lorentz transformation. The period T can also be generalized as a measure of t. Extrapolating from the first- through third-order Lorentz transformations, a general form of the nth-order Lorentz transformation is proposed. For the first through third orders v, a, and j

$$\gamma_v = \sqrt{1-\frac{v^2}{c^2}} \tag{81}$$

$$\gamma_a = \sqrt{1-2\frac{a\cdot d}{c^2}} \tag{82}$$

$$\gamma_j = \sqrt{1-6\frac{j\cdot d\cdot t}{c^2}} \tag{83}$$

and a zeroth-order d transformation (the null transformation)

$$\gamma_d = \sqrt{1 - \frac{0}{c^2}} \tag{84}$$

is relatively safe to propose. Keeping in mind that the determinant must be dimensionless, the general form of the nth-order Lorentz transformation then appears to be

$$\gamma_n = \sqrt{1 - n! \frac{\left(\frac{v}{\mathbf{t}^{n-1}}\right) \cdot d \cdot t^{n-2}}{c^2}}. \tag{85}$$

E. The origin and the axes

If the worldlines of all points on or in an object have the same angular velocity about an axis within that same object, the object is said to have spin. Angular velocity, frequency, and spin are synonymous dimensions. They are all related to a centripetal acceleration, which in turn is synonymous with linear acceleration. The linear velocity component v_θ of any specific point on or in a spinning object varies with its distance from the axis of spin, but the frequency, angular velocity, and spin are constant regardless of position.

Einstein proposed that theoretical consideration of linear acceleration gives information about gravitational acceleration.[53] This proposition is known as the principle of equivalence. Because the angular velocity, frequency, and spin are related to a centripetal acceleration, which in turn is synonymous with linear acceleration, the principle of equivalence extends to all of these cases. There appears to be no reason to limit this extension to the dimension of acceleration. One linear velocity, irrespective of magnitude, is just as good as any other; any one force is just as geometrically predictable as the rest. Therefore, a more universal statement is proposed as a fundamental property of the Dimensional Coordinate System: theoretical consideration of the general dimensional qualities of any physical process gives information about any synonymous dimensions.

Linear and gravitational acceleration are synonymous dimensions, and all worldlines for acceleration are similar functions of second order in the space-time plane. Angular velocity and frequency are synonymous dimensions, and all worldlines for angular velocity and frequency are similar functions. Any two units that express synonymous dimensions also express similar functions, i.e., years and days, meters and yards, joules and kelvins. Geometrically, this means that all second-order worldlines in the space-time plane are equivalent to acceleration, and therefore the same results are predicted for linear or gravitational acceleration.

The geometric description of spin S is that of angular velocity ω, the sine curve depicted in Fig. 7(c) and given in Eq. (56). Qualitatively then, d as a function of spin, in its most basic form, is given by the expression for acceleration in Eq. (34). The physical quantity spin describes the case of true uniform circular motion. Quantitatively, with its four permutations of positive and negative, acceleration and deceleration, Eq. (35) is all that is needed to express spin. In actual practice, spin is commonly expressed as an integer proportion, from the set of rational numbers $\mathbb{Q}$, of Planck's physical constant $\hbar$.

The dimensions of $\hbar$ are those of angular momentum $\mathbf{L}$. These dimensions include factors of mass and of specific angular momentum, the inverse of acceleration. The physical direction of $\mathbf{L}$ is perpendicular to both the t- and d-axes of the worldline of any point on or in any object with spin. This perpendicularity does not necessarily exist for an object free in space. The spin axis of a free object can have a nonperpendicular orientation to its velocity axis, and therefore to its t-axis. This quantitatively distinguishes a free object from free space-time, and one free object from another, as applied by Pauli in his exclusion principle. Thus the $\mathbf{L}$-axis of a free object is parallel (or antiparallel) to its m-axis.

The geometric description of a free object in the DCS is that of the origin of a unique dimensional triad (t, d, m). Qualitatively, a free object has the dimensionless measure of number N. Quantitatively, N is real and can therefore be counted, is nonzero and positive, and is given by the closed set of all natural numbers $\mathbb{N}$. Large numbers of N may be expressed in proportion to a standard quantity n. The SI quantity of n is the mole 6.0×10^{23} N. Furthermore, when quantities with similar dimensional qualities are expressed in proportion to each other, the ratio is dimensionless. Qualitatively, n is any dimensionless quantity. Some synonymous dimensions are the radian and the gross. Quantitatively, n is given by the set of all real numbers $\mathbb{R}$ (which has $\mathbb{N}$ as a subset).

The dimensions of the DCS axes need not be limited to the triad (t, d, m). Any triad consisting of axes related by the same proportions as $ct = d$, and dc^2/G, such as (v, v^2, F) or $(a, d^2/t^3, md/t^3)$, will have the same geometric relations as a triad of the prime dimensions.

The scaling of DCS axes is not limited to linear incrementation: axes of second and greater order scaling create curled up dimensional spaces; allowing the orientation of coordinate axes to vary from the perpendicular creates Gaussian curved spaces. As discussed in Sec. II D, all methods applicable to the geometric space (x, y, z) are equally applicable to the dimensional space (t, d, m). One such interesting scale is the exponentially scaled triad (e^t, e^d, e^m) where e is the base of the natural logarithm. The Taylor series expansion of the form

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{86}$$

gives the discrete components

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{87}$$

for each respective DCS axis. The first term 1 is not only x^0, but also, both qualitatively and quantitatively, the origin N in the DCS as the point of intersection $t^0 = d^0 = m^0 = 1 = N$. The factor $1/n!$ should be familiar from the discussion in Sec. III C and its application to the Lorentz transformation in Sec. III D. The remaining factor x^n is the subject of the following section.

IV. THE PERIODIC TABLE OF DIMENSIONS

The idea is that, armed with the fundamental and derived dimensions, every numerically expressible quantity in the universe can be formulated. Carried further, this means that every possible relation, encompassing every facet of our knowledge about the physical world can be—at least in theory—uniquely defined.

-T. Szirtes[4]

A. The elements

Consider a system composed of three exponentially scaled, mutually perpendicular axes t, d, and m, with unit increments of t^N, d^N, and m^N, respectively. Within this small domain the dimensional terms for time t^1, space d^1, and mass m^1 are easily identified. Other terms such as area $A = d^2$ and volume $V = d^3$ also appear as derived dimensions on these simple x^N number lines. Applying the $\pm$ operator to the exponent allows the representation of further components, such as frequency $f = t^{-1}$, as a reflection across the origin.

Now consider a coordinate system constructed of all points $(t^{\pm n}, d^{\pm n}, m^{\pm n})$ when n is allowed to range freely over all real values for each axis. Combination of the terms $t^{\pm n}$, $d^{\pm n}$, and $m^{\pm n}$ allows the representation of every possible derived combination of mass, space, and time as a product $m^{\pm n}d^{\pm n}t^{\pm n}$ (the order of the variables t, d, and m is reversed when expressing a product to help avoid confusion).[5] One version of Occam's razor wisely advises that "entities should not be multiplied unnecessarily." In *The Harmony of the World* I XX, Kepler advises that "No operation of addition or subtraction gives rise to diversity, but all are equally related to their pair of 'terms' or 'elements.'"[59] Algebraically, what Kepler is saying is that unlike terms cannot be added—they can only be multiplied, and multiplication derives diversity. Alternatively, Occam is saying that multiplication should be used sparingly, so it is also prudent to set some upper limit on n to keep it real. After all, physical quantities are measurements of real events.

Szirtes and Jackson both examine the question of how many dimensions a dimensional system should have.[4,60] Szirtes demonstrates the possibility, and impracticality, of both mono- and omni-dimensional systems. In a one-dimensional system everything is derived and there is therefore no resolution, i.e., differentiation, between the units; in an infinitely-dimensional system everything is fundamental and therefore relates no common traits among the components. He concludes by recommending the SI system with its seven fundamental dimensions: length, time, mass, electric current, amount of substance, temperature, and luminous intensity. However, there are ambiguous and redundant physical qualities expressed by some fundamental SI units. Amount of substance n is dimensionless, and temperature can be derived as energy with the dimensions md^2t^{-2}. Electric current and luminous intensity can also be resolved in terms of $t^{\pm n}$, $d^{\pm n}$, and $m^{\pm n}$, so that the entire SI system may be resolved into a system of three primary dimensions t, d, and m, from which all other dimensions $m^{\pm n}d^{\pm n}t^{\pm n}$ may be derived.[54]

TABLE II: The System of Prime Dimensions (SP).

SP dimension	SI unit
number (n)	mole (mol)
polarity ($\pm$)	n/a
time (t)	second (s)
space (d)	meter (m)
mass (m)	kilogram (kg)

A system of prime dimensions, derived from SI units, establishes the fundamental relationships from which all other physical qualities can be derived. The System of Prime Dimensions (SP) shown in Table II consists of the primary natural qualities described by the DCS axes: mass, space, and time. To quantitatively evaluate the results of any geometric analysis, two more physical characteristics are needed: one to express magnitude, and one to express polarity. These are the scalar n and the arithmetic operator $\pm$. Neither n nor $\pm$ are dimensions. They are included strictly to facilitate quantitative analysis.

Every quality of nature can be derived as products of these prime dimensions. If every derived combination of the three prime dimensions represented by the DCS axes is considered, the number of possible derivations is infinite. However, there are practical limits to the number of measurements necessary to conveniently describe every aspect of nature.

After considering the number of dimensions, Szirtes turns to the question of standard magnitudes. Since all quantities go to n, SP is not a system of units. It is a system of dimensional relations with all quantity removed, and thus any system of units may be applied; SI is the present standard. Both SI and SP are coherent systems of dimensions, in that the derived dimensions are expressed as combinations of the fundamental dimensions without conversion factors other than 1.[4,61]

B. The table

The construction of a three-axis periodic table, based on the prime dimensions of the DCS axes, and illustrated in Fig. 9, provides a visual medium for an investigation of derived dimensions. Each axis in this Periodic Table of Dimensions has both positive (+) and negative (−) coordinates, 180 degrees in opposition from the origin; each coordinate represents an exponent n. Positive coordinates express positive exponents in the numerator, while negative coordinates express positive exponents in the denominator, according to convention. Expressing a derived dimension as a product of factors with either positive or negative exponents is equally valid. Unity, represented by the unit scalar N, is placed at the origin. Each step away from the origin along any axis represents an exponentially incremented, or decremented, factor of the dimension assigned to that axis. Linear increments of the exponent of each prime dimension proceed along each axis. Just as in any other three-axis coordinate system, every coordinate triplet (t, d, m) represents a unique position; in the Periodic Table of Dimensions each unique position represents a unique physical dimension.

As a table in three directions, the Periodic Table of Dimensions is best viewed in 3-D rather than on a 2-D page or computer screen. An interactive html table is available online at EPAPS, but a 3-D construction makes an excellent classroom tool for presentations.[62] Abbreviated versions are frequently sufficient. For example, a discussion on Newtonian mechanics requires no more than the lower left-hand quadrant of the table presented here, while the ideal gas law requires approximately one-third of the table to include both pressure and volume. Construction drawings are also available at EPAPS.

C. Dimensional resolutions

The universal principle of equivalence provides the theoretical basis to resolve every numerically expressible quantity in the universe to the prime dimensions of time, space, and mass. The universal principle of equivalence is demonstrated beyond general relativity by the superposition of forces and the superposition of energies. Mathematically, superposition is demonstrated by addition of like terms. Addition and subtraction do not alter the dimensions of any physical equation, but multiplication and division provide dimensionally diverse results. This general principle describes the limited diversity necessary to describe any derived dimension. Rather than requiring a separate description for accelerations caused by nonuniform linear translation, uniform rotation, and gravitation, it is only necessary to describe the general case of acceleration, to which a scalar is applied for the description of the specific case; an expression of polarity can be used to further differentiate, i.e., deceleration vs. acceleration. This is true of every dimensional quality of nature, including forces and energies.

In an empirical attempt to qualify the claim that every numerically expressible quantity in the universe can be formulated in the periodic table, 1,727 unique physical measurements have been identified at the time of this writing in a limited search including standard references.[4,63–65] Of the unique physical measurements identified, 1,626 (94.2%) are defined by sources in terms of fundamental units of time, space, mass, and electric charge. Only those measurements with clear definitions by outside sources in terms of the prime dimensions or electric charge and those used in the context of this paper were considered. Once resolved to prime dimensions, each measurement is cataloged with its synonymous dimensions. This catalog is also available online at EPAPS.[62] A representative sample demonstrating the statistical extremes and the diversity of the measurements, units, and sources is presented in Table III. In general, web resources were avoided. Russ Rowlett's Dictionary of Units of Measurement is a notable exception.[65]

Certainly, the fact that "dozen" is defined in only one source simply reflects the lack of cookbooks in the survey. A definite bias towards physics has been exercised, but the citation of the dimensions of thermal conductivity 30 times seems to be statistically significant. This may be due in part to a consensus on nomenclature; not one other dimension synonymous with thermal conductivity was identified. In all, the number of times a measurement is defined according to the standards of this survey has little bearing on the results beyond the lack of a supporting reference in 19 cases.

There are 72 unique resolutions cataloged by SP dimensions at the time of this writing. All are within the limits of t^{-6} to t^3, d^{-5} to d^4, and m^{-2} to m^1. The 66 physical quantities within the limits of t^{-4} to t^2, $d^{\pm 4}$, and $m^{\pm 1}$ fit inside the $7 \times 9 \times 3$ rectangular volume of 189 possible triplet combinations presented in Fig. 9. Many of the measurements are dimensional inversions of other derived dimensions. A simple example is frequency, whose inverse is time. Elimination of these redundancies would reduce the number of unique dimensions, but would promote ambiguity. Furthermore, no clear axial symmetry of inversion is apparent within the scope of identified dimensions. There is some symmetry about the axis in the space-time plane containing velocity dt^{-1} and specific energy d^2t^{-2}, and even less about an approximate axis through all three planes including fluidity $m^{-1}d^1t^1$ and dynamic viscosity $md^{-1}t^{-1}$, but there is no one-to-one correspondence. Many of the possible derivations have no associated measurements yet, but just as with the Periodic Table of Elements, applications of the yet-to-be-filled elements of the Periodic Table of Dimensions may just be waiting to be discovered.

One such application is that of the physical constant. When the Planck units are applied to the elements of this table, the magnitudes of the derived dimensions are physical constants. By definition, t, d, and m in Planck units are the Planck time t_{P}, the Planck length d_{P}, and the Planck mass m_{P}. From these the derived magnitude

The Periodic Table of Dimensions

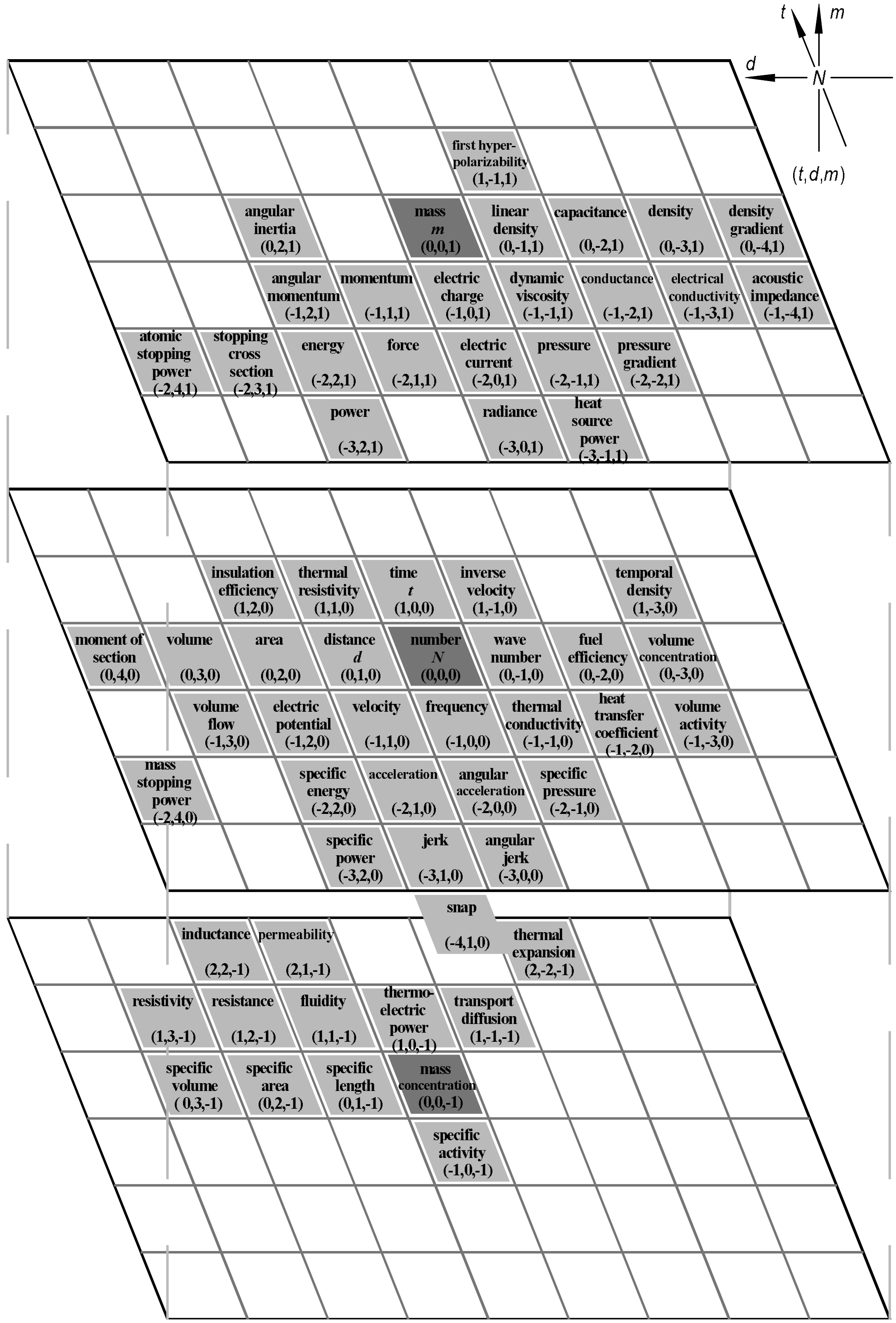

FIG. 9: The Periodic Table of Dimensions. Three space-time coordinate planes, graduated from top to bottom as the m, N, and m^{-1} planes, respectively, give an ordered presentation of every measurement that can be derived from the prime dimensions of mass m, space d, and time t.

TABLE III: A representative summary of the Catalog of Synonymous Dimensions.

Sample measurement	Number of citations	Fundamental quantity	SI unit	SP dimensions	Synonymous dimensions
dozen	1	number	mole	n	213
yard	3	distance	meter	d	209
gallon	3	volume	cubic meter	d^3	203
pound-mass	4	mass	kilogram	m	131
acre	3	area	square meter	d^2	105
torque	4	energy	newton-meter	md^2t^{-2}	96
half-life	4	time	second	t	80
bar	5	pressure	pascal	$md^{-1}t^{-2}$	72
rpm	2	frequency	hertz	t^{-1}	42
luminous intensity	6	power	candela	md^2t^{-3}	32
electric charge[a]	5	mass flow	coulomb	mt^{-1}	13
surface tension	11	surface tension	newton per meter	mt^{-2}	13
thermal conductivity	30	thermal conductivity	watt per meter-kelvin	$d^{-1}t^{-1}$	1

[a] A discussion on the dimensions of electric charge follows.

of the velocity element $d_\mathrm{P}t_\mathrm{P}^{-1}$ is found to be light speed c, the derived magnitude of the angular momentum element $m_\mathrm{P}d_\mathrm{P}^2t_\mathrm{P}^{-1}$ is found to be Planck's reduced constant $\hbar$, and the element $m_\mathrm{P}^{-1}d_\mathrm{P}^3t_\mathrm{P}^{-2}$, which had no corresponding measurements in the survey, is found to be the gravitational constant G. Still other elements are derivations of these constants. The derived magnitude of the mass flow element $m_\mathrm{P}t_\mathrm{P}^{-1}$ is c^3/G. Since the Planck units themselves are derived from the constants c, G, and $\hbar$ alone, it can be expected that all further derivations will be functions of some order of each.

Of all the dimensional resolutions examined in this survey, electric charge q stands out along with the prime dimensions as essentially unresolved. This seems to result from an arbitrary emphasis on electric current q/t as a fundamental unit.[60] However, charge is not without its derivations as a product of prime dimensions. The *CRC Handbook* discusses three variants of the cgs system: the electrostatic, the electromagnetic, and the Gaussian systems.[61] In the electrostatic system, q has the dimensions $m^{1/2}d^{3/2}t^{-1}$; in the electromagnetic system q/t is defined with the dimensions $m^{1/2}d^{1/2}t^{-1}$ so that q has the dimensions $m^{1/2}d^{1/2}t^{0}$ and the product qc has the dimensions $m^{1/2}d^{3/2}t^{-1}$; the Gaussian system is a hybrid of these two. These dimensions of q are confirmed by Allen as $m^{1/2}d^{3/2}t^{-1}\epsilon^{1/2}$, where the permittivity ϵ is dimensionless.[64] When the SP dimensions of ϵ, which are m/d^3, synonymous with density, are taken into account, $m^{1/2}d^{3/2}t^{-1}\epsilon^{1/2}$ becomes m/t.

In order to derive the $m^{1/2}d^{3/2}t^{-1}\epsilon^{1/2}$ dimensions, two fundamental electromagnetic constants are left dimensionless: the permeability μ_o and permittivity ϵ_o of free space. Furthermore, factors of π appear where they are not expected or wanted in equations using these systems. This is an example of a nonrationalized system. A rationalized system includes factors of π in the constants to remove the unwanted factors of π from the results.[60,61,64] The SI and SP systems are rationalized systems in that μ_o is defined with a factor of 4π, ϵ_o is defined as a function of μ_o, and their proportion is determined by the speed of light in the relation

$$\mu_o\epsilon_o c^2 = 1. \tag{88}$$

It is in the dimensions of q that SI and SP depart, and in SP q is derived from the fundamental electromagnetic relationship given in Eq. (88). In SI q/t is a fundamental dimension, and the unit of inductance, the henry, has the dimensions[4,61] $md^2t^{-2}(q/t)^{-2}$. Permeability has the dimensions of henry per meter, which resolve to $mdt^{-2}(q/t)^{-2}$, in which q is the only dimensionally ambiguous factor in SP terms.

In SP, the derivation of q follows from the SI definition of the fundamental unit of electric current:

> *The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newtons per meter of length. (9th CPGM [1948], Resolutions 2 and 7).*

The measurable qualities in this definition of electric current are force per distance. Resolved to the fundamental components of the DCS axes, force per distance becomes mt^{-2}. All other electric quantities can be derived from electric current. In SI units, the ampere-second comprises the fundamental unit of charge, the coulomb. In DCS terms, the geometric description of q becomes a line with slope m/t. In the Periodic Table of Dimensions, this makes q synonymous with mass flow, as opposed to the electrostatic system definition with fractional exponents.

Given m/t as the dimensions of q, the dimensions of μ_o become dt^2/m, the reciprocal of pressure P. From Eq. (88), and with c and μ_o defined dimensionally, ϵ_o has dimensions synonymous with density ρ. Rearrangement of Eq. (88) gives the relation $c = (1/\mu_o\epsilon_o)^{1/2}$. Since μ_o is synonymous with the reciprocal of P and ϵ_o is synonymous with ρ, the speed of light is then qualitatively defined by the relation $c \propto (P/\rho)^{1/2}$, which is synonymous with the theoretical formula for the speed of sound in air,[66,67] $v_{\mathrm{sound}} = (1.4P_{air}/\rho_{air})^{1/2}$. Furthermore, the electric field $\mathbf{E}$ is defined[60,68] as the force per unit charge, $\mathbf{E} = F/q$. This has the SP dimensions of velocity. Analogous to $\mathbf{E}$ is the gravitational field $\mathbf{G}$, derived as the force per unit mass, $\mathbf{G} = F/m$. This has the SP dimensions of acceleration. This gives the analogous result in both systems that the force is a product of the source and the field:

$$F(\mathbf{G}) = ma, \tag{89a}$$

$$F(\mathbf{E}) = qv. \tag{89b}$$

The qualitative similarity between field equations and boundary conditions in different physical contexts and the SI definition of electric current are strong grounds for favoring the use of m/t over $m^{1/2}d^{3/2}t^{-1}$ as the dimensions of q.

D. Dimensional arithmetic

The table not only has organizational properties, but also mathematical properties. In the Periodic Table of Dimensions (Fig. 9), mathematical operations on dimensions cause movement through the table in proportion to the dimensions of the terms; all of these operations are the result of standard exponential convention. Addition and subtraction are limited to like terms. The addition or subtraction of like terms results in no change to the dimensions of the terms and therefore no movement through the table—no operation of addition or subtraction gives rise to diversity. As an example, 5 meters (dimension d) can be added to 1 meter (dimension d) and the dimension of the result, 6 meters, is still d.

Multiplication of any two dimensions proceeds from one factor, by addition of the coordinates (the exponents) of the second factor, to the product. Dimensional multiplication is illustrated in Fig. 10(a). For Newton's second law $F = ma$, force is arrived at by adding the position of m [1 up (from the origin)] to the position of a [2 forward, 1 left (from the origin)]. On paper this is accomplished arithmetically by addition of the individual components of the dimensional triad, i.e.,

$$\begin{array}{ccccc} F & = & m & \times & a \\ (-2,1,1) & = & (0,0,1) & + & (-2,1,0). \end{array} \tag{90}$$

Division is simply the inverse operation proceeding from the numerator. Rearranging the ideal gas law for pressure

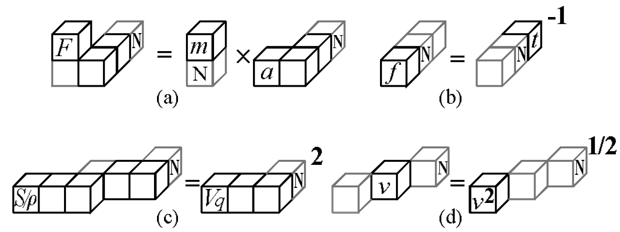

FIG. 10: Dimensional arithmetic. Each cube represents an element in The Periodic Table of Dimensions: (a) Force F is derived from the multiplication of mass m and acceleration a, (b) frequency f is the inversion of time t, (c) mass stopping power S/ρ is derived by squaring electric potential V_q, and (d) velocity v is derived as the square root of specific energy v^2.

gives $P = nRT/V$, where n and R are dimensionless so that their product is the origin N. Then nR times the temperature T is synonymous with energy E, whose position is 2 forward, 2 left, 1 up; volume V is 3 left. For multiplication, movement is away from the origin; for division, movement is toward the origin. Then for T/V, the coordinates of V are subtracted (move 3 right instead of left) from those of E (2 forward, 2 left, 1 up) to arrive at P (2 forward, 1 right, 1 up). Arithmetically this is accomplished as

$$\begin{array}{ccccccc} P & = & nR & \times & T & \div & V \\ (-2,-1,1) & = & (0,0,0) & + & (-2,2,1) & - & (0,3,0). \end{array} \tag{91}$$

Inversion is realized as a symmetric translation across the origin and is illustrated in Fig. 10(b). Frequency f with position 1 forward is the inversion of time t, 1 back; for 3 left it is 3 right. For permeability at 2 back, 1 left, 1 down it is pressure (2 forward, 1 right, 1 up). For those who think better in coordinate notation than in shapes, just multiply the dimensional triad by -1, i.e.,

$$\begin{array}{ccc} \mu & = & P^{-1} \\ (2,1,-1) & = & -1 \times (-2,-1,1). \end{array} \tag{92}$$

Powers of any dimension are multiples of that position. For velocity v with position 1 forward, 1 left, the square is v^2 at 2 forward, 2 left. Figure 10(c) shows that mass stopping power S/ρ (2 forward, 4 left) is derived by squaring electric potential V_q (1 forward, 2 left), or as an arithmetic operation

$$\begin{array}{ccc} S/\rho & = & V_q^2 \\ (-2,4,0) & = & 2 \times (-1,2,0). \end{array} \tag{93}$$

Roots of any dimension are fractions of that position; v at 1 forward, 1 left, is the square root of v^2 (2 forward, 2 left) as shown in Fig. 10(d), or arithmetically

$$\begin{array}{ccc} v & = & \sqrt{v^2} \\ (-1,1,0) & = & \frac{1}{2} \times (-2,2,0). \end{array} \tag{94}$$

The cube root of volume (3 left) is distance (1 left). The geometric mean of dimensions is the root of their product. This is given as the geometric center of their positions in the table. For energy (2 forward, 2 left, 1 up) and inductance L (2 back, 2 left, 1 down), the geometric mean is area (2 left); 2 forward and 2 back cancel out, and 1 up and 1 down cancel out. In coordinate terms this is

$$\begin{aligned} (E \quad \times \quad L)^{1/2} \quad = \quad A \\ \frac{1}{2}[(-2,2,1)+(2,2,-1)] = (0,2,0). \end{aligned} \tag{95}$$

The geometric mean of the left hand side of Eq. (88) is the origin. This can also be proven by constructing the triangle μ, ϵ, and v^2, and noting the intersection of the lines originating from each apex and bisecting the side opposite. Question: Is this intersection the square root or the cube root?

V. CONCLUSIONS

What is a dimension? A dimension is any measurable physical quantity. From the simple space metric given by the Pythagorean distance formula in Eq. (1), to the ordered Periodic Table of Dimensions presented in Fig. 9 which demonstrates geometric relations between physical phenomena, this discussion has been guided primarily by the principles of geometry. By contracting the space metric to the only kinematically relevant direction, that of velocity, space and time can be expressed in equivalent terms. The relations of these terms can be operated on by the space-time metric in any of its forms. As independent quantities expressed in like terms, space and time form an orthogonal plane. Detailed geometric analyses of kinematic functions in the space-time plane accurately predict real physical phenomena. With the addition of only one more measurable physical quantity, mass, orthogonal to the space-time plane, a geometric representation of over 1600 derived units reduced to 72 elements brings to light a geometry of dimensional analysis. Each of the elements is a measurable physical quantity derived from the three primary axial quantities mass, space, and time; all physical law can be derived from these elements with geometric principles.

And direction? No, it does not imply dimension. "Dimensioned" is used as a verb when a technical drawing is dimensioned in each direction. Under close scrutiny, directions are found to combine in a different proportion than do the primary physical qualities from which all physical law can be derived. When dimension is used as a noun to refer to measurable physical quantities, direction does not measure up to space, time, or mass. Then in a formal discussion of direction versus dimension, it must be said that direction does not imply dimension.

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[1] E. A. Abbott, *Flatland* (Dover, New York, NY, 1992), 2nd ed.: Preface, p. viii; Part II, Sec. 16, p. 58.

[2] H. Minkowski, "Space and time," Cologne (1908); from A. Einstein, H. A. Lorentz, H. Minkowski, and H. Weyl, *The Principle of Relativity*, translated by W. Perrett and G. B. Jeffery with notes by A. Sommerfeld (Dover, New York, NY, 1952), pp. 73-91. It has been noted by A. A. Martinez, Ref. 36, that the translations by Perrett and Jeffery have "some slight but significant defects."

[3] C. F. Bohren, "Dimensional analysis, falling bodies, and the fine art of *not* solving differential equations," Am. J. Phys. **72**, 534-537 (2004).

[4] T. Szirtes, *Applied Dimensional Analysis and Modeling* (McGraw-Hill, San Francisco, CA, 1998), Sec. 3.

[5] J. F. Price, "Dimensional analysis of models and data sets," Am. J. Phys. **71**, 437-447 (2003).

[6] H. C. Ohanian, *Gravitation and Spacetime* (Norton, New York, NY, 1976), Sec. 2.

[7] R. J. Cook, "Physical time and physical space in general relativity," Am. J. Phys. **72**, 214-219 (2004).

[8] A. Einstein, "On the electrodynamics of moving bodies," Ann. Phys. (Leipzig) **17**, (1905); from the English translation in Ref. 2, pp. 35-65.

[9] A. Einstein, *Relativity: The Special and General Theory* (Crown, New York, NY, 1961).

[10] A. Einstein, "The foundation of the general theory of relativity," Ann. Phys. (Leipzig) **49**, (1916); from the English translation in Ref. 2, pp. 109-164.

[11] H. Weyl, "Gravitation and electricity," Sitzungsberichte der Preussischen Akad. d. Wissenschaften (1918); from the English translation in Ref. 2, pp. 199-216.

[12] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, NY 1975), 2nd ed., Sec. 11.

[13] J. A. Rhodes, and M. D. Semon, "Relativistic velocity space, Wigner rotation, and Thomas precession," Am. J. Phys. **72**, 943-960 (2004).

[14] R. M. Wald, "Resource letter TMGR-1: Teaching the mathematics of general relativity," Am. J. Phys. **74**, 471-477 (2006).

[15] R. M. Jonsson, "Visualizing curved spacetime," Am. J. Phys. **73**, 248-260 (2005).

[16] Y. Pierseaux, "The 'fine structure' of Special Relativity and the Thomas precession," Ann. de Broglie **29**, 57-118 (2004).

[17] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, CA, 1973), Sec. 1.

[18] E. G. P. Rowe, "Rest frames for a point particle in special relativity," Am. J. Phys. **64**, 1184-1196 (1996).

[19] C. Criado, and N. Alamo, "A link between the bounds on relativistic velocities and areas of hyperbolic triangles," Am. J. Phys. **69**, 306-310 (2001).

[20] M. W. P. Strandberg, "Special relativity completed: The source of some 2s in the magnitude of physical phenomena," Am. J. Phys. **54**, 321-331 (1986).

[21] J. D. Hamilton, "Relativistic precession," Am. J. Phys. **64**, 1197-1201 (1996).

[22] A. Einstein, "Cosmological considerations on the general

theory of relativity," Sitzungsberichte der Preussischen Akad. d. Wissenschaften, (1917); from the English translation in Ref. 2, pp. 175-188.
[23] G. H. Goedecke, "Geometry of the Thomas precession," Am. J. Phys. **46**, 1055-1056 (1978).
[24] E. Minguzzi, "The Minkowski metric in non-inertial observer radar coordinates," Am. J. Phys. **73**, 1117-1121 (2005).
[25] Reference 17, Sec. 2.
[26] P. Aravind, "Temporal order in 1+1-dimensional relativity," Am. J. Phys. **56**, 258-262 (1988).
[27] N. D. Mermin, "An introduction to space-time diagrams," Am J. Phys. **65**, 476-486 (1997).
[28] D. Brill, and T. Jacobson, "Spacetime and Euclidean geometry," Gen. Relativ. Gravit. **38**, 643-651 (2006).
[29] Reference 17, Sec. 6.
[30] N. D. Mermin, *Space and Time in Special Relativity* (McGraw-Hill, San Francisco, CA, 1968), Ch. 17.
[31] A. Einstein, "Hamilton's principle and the general theory of relativity," Sitzungsberichte der Preussischen Akad. d. Wissenschaften, (1916); from the English translation in Ref. 2, pp. 165-173.
[32] Reference 6, Sec. 6; Ref. 17, Sec. 38.4.
[33] T. M. Helliwell and D. A. Konkowski, "Cosmic strings: Gravitation without local curvature," Am. J. Phys. **55**, 401-407 (1987).
[34] Reference 6, Sec. 8; Ref. 17, Sec. 31.2.
[35] R. Ferraro, "The equivalence principle and the bending of light," Am. J. Phys. **71**, 168-170 (2003).
[36] J. P. Costella, B. H. J. McKellar, A. A. Rawlinson, and G. J. Stephenson, Jr., "The Thomas rotation," Am. J. Phys. **69**, 837-847 (2001); A. A. Martinez, "Kinematic subtleties in Einstein's first derivation of the Lorentz transformations," Am. J. Phys. **72**, 790-798 (2004).
[37] Reference 30, Chs. 4-6.
[38] R. A. Muller, "Thomas precession: Where is the torque?" Am. J. Phys. **60**, 313-317 (1992).
[39] Reference 17, Sec. 27.3.
[40] A. Sommerfeld, "Notes [on Minkowski's Space and time]"; from the English translation in Ref. 2, pp. 92-96.
[41] J. Silk, A. S. Szalay, and Y. B. Zel'dovich, "The large scale structure of the universe" (1983), from Scientific American, *Particle Physics in the Cosmos*, R. A. Carrigan, Jr., and W. P. Trower eds. (W. H. Freeman, New York, NY, 1989), Ch. 3.
[42] S. Hawking, *A Brief History of Time* (Bantam Books, New York, NY, 1988), Chs. 2, 3; J. D. Barrow and J. Silk, "The structure of the early universe" (1980), Ref. 41, Ch. 2; F. Wilczek, "The cosmic asymmetry between matter and antimatter" (1980), Ref. 41, Ch. 10.
[43] Reference 17, Sec. 7.3.
[44] L. M. Krauss, "Dark matter in the universe" (1986), Ref. 41, Ch. 1; D. A. Dicus, J. R. Letaw, D. C. Teplitz, and V. L. Teplitz, "The future of the universe" (1983), Ref. 41, Ch. 12; Ref. 30, Ch. 16.
[45] H. C. Ohanian, *Physics*, 1st ed. (W. W. Norton, New York, NY, 1985), Ch. 2.
[46] Reference 45, Ch. 17.
[47] N. D. Mermin, "Space-time intervals as light rectangles," Am J. Phys. **66**, 1077-1080 (1998).
[48] T. R. Sandin, "The Jerk," Phys. Teach. **28**, 36-40 (1990).
[49] S. H. Schot, "Jerk: The Time Rate of Change of Acceleration," Am. J. Phys **46**, 1090-1094 (1978).
[50] J. C. Sprott, "Some Simple Chaotic Jerk Functions," Am. J. Phys. **65**, 537-543 (1997); K. E. Chlouverakis, and J. C. Sprott, "Chaotic Hyperjerk Systems," Chaos, Solitons, & Fractals **28**, 739-746 (2005); M. Visser, "Jerk, Snap, and the Cosmological Equation of State," Class. Quantum Grav. **21**, 2603-2615 (2004).
[51] K. Mehrotra, and P. R. Mahapatra, "A Jerk Model for Tracking Highly Maneuvering Targets," IEEE Trans. Aerospace and Electron. Sys. **33**, 1094-1105 (1997).
[52] Reference 9, appendix III.
[53] A. Einstein, "On the influence of gravitation on the propagation of light," Ann. Phys. (Leipzig) **35**, (1911); from the English translation in Ref. 2, pp. 97-108.
[54] Reference 6, Sec. 5.
[55] A Einstein, "Explanation of the perihelion motion of Mercury from the general theory of relativity," Sitzungsberichte der Preussischen Akad. d. Wissenschaften (1915); from *The Collected Papers of Albert Einstein, vol. 6, The Berlin Years: Writings, 1914-1917*, (Princeton university press, Princeton, NJ, 1997) pp. 112-116. Translation by B. Doyle, reprinted from K. R. Lang, and O. Gingerich, eds., *A Sourcebook in Astronomy and Astrophysics, 1900-1975* (Harvard University press, 1979).
[56] H. C. Corben, "Factors of 2 in magnetic moments, spin-orbit coupling, and Thomas precession," Am. J. Phys. **61**, 51-52 (1993).
[57] L. I. Schiff, "On experimental tests of the general theory of relativity," Am. J. Phys. **28**, 340-343 (1960).
[58] Reference 17, box 40.4.
[59] J. Kepler, *The Harmony of the World*, translated by E. J. Aiton, A. M. Duncan, and J. V. Field (The American Philosophical Society, Philadelphia, PA, 1997), Book I, Definition XX.
[60] Reference 12, Appendix. In this reference Jackson also recommends on the subject of units and dimensions: M. Abraham, and R. Becker, *Electricity and Magnetism* (Blackie, London 1937); R. T. Birge, Am. Phys. Teacher (now Am. J. Phys.) **2**, 41 (1934), **3**, 102, 171 (1935); P. W. Bridgman, *Dimensional Analysis* (Yale University Press, 1931); M. Planck (no specific reference provided).
[61] E. R Cohen, and P. Giacomo, "Symbols, Units, and Nomenclature in Physics," *CRC Handbook of Chemistry and Physics* (CRC Press, Boca Raton, FL, 1989), 70th ed., pp. F296-F341.
[62] These materials are deposited at EPAPS Document No. ***. This document may be retrieved via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html) or from ftp.aip.org in the directory /epaps/. See the EPAPS homepage for more information.
[63] E. R. Cohen, *The Physics Quick Reference Guide*, (AIP Press, Woodbury, NY, 1996); R. C. Weast and D. R. Lide, eds., *CRC Handbook of Chemistry and Physics* (CRC Press, Boca Raton, FL, 1989), 70th ed.; E. A. Avalon and T. Baumeister III, eds., *Mark's Standard Handbook for Mechanical Engineers* (McGraw-Hill, San Francisco, CA, 1978) 9th ed.; E. Oberg, F. D. Jones, and H. L. Horton, *Machinery's Handbook* (Industrial Press, New York, NY, 1988) 23rd ed.
[64] C. W. Allen, *Astrophysical quantities* (Oxford University Press, New York, NY, 1963), 2nd ed., pp. 28-29.
[65] R. Rowlett, "How many? A Dictionary of Units of Measurement," www.unc.edu/∼rowlett/units (2003).
[66] Reference 64, p. 114.
[67] Reference 45, Ch. 16.2.
[68] Reference 45, Ch. 23.